Penrith Selective
High School

2011
Higher School Certificate
Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question
- All questions are of equal value
- Staple this test to your answers
- Attempt Questions 1 – 7

Total marks – 84

Question	Mark
1	/12
2	/12
3	/12
4	/12
5	/12
6	/12
7	/12
Total	/84
Percentage	

Student's Name:______________________________

Teacher:______________________________

Question 1 (Start a new page). **(12 marks).**

a) Find the coordinates of the point which divides the interval joining (-3,5) to (8,4) in the ratio 1:3 **(2)**

b) Find the equations of the two lines inclined at 45° to the line $3x + y = 8$ and passing through the point $A(4,0)$. **(3)**

c) Find $\int x\sqrt{3-x}dx$, using the substitution $u = 3 - x$ **(3)**

d) Solve $\frac{2x+5}{x} \geq 1$ **(2)**

e) Without using a calculator, show that $sin\left[cos^{-1}\left(\frac{3}{5}\right) + tan^{-1}\left(-\frac{3}{4}\right)\right] = \frac{7}{25}$ **(2)**

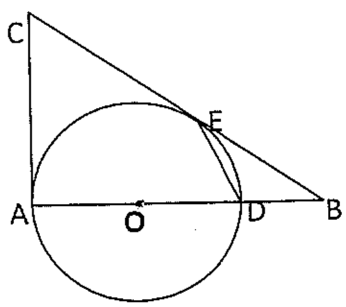
Question 2 (Start a new page). **(12 marks).**

a) Find the Cartesian equation of the curve and describe it geometrically if **(3)**
$x = \sin\theta + \cos\theta$ and $y = \sin\theta - \cos\theta$

b) i) Derive the equation of the normal to $x^2 = 4ay$ at $(2ap, ap^2)$. **(3)**

ii) The chord joining $P(2ap, ap^2)$ to $Q(2aq, aq^2)$ passes through $T(0, -2a)$ show that $pq=2$. **(3)**

c) *AD* is a diameter of the circle centre *O*. *E* is a point on the circumference. **(3)**
Tangents at *A* and *E* meet at *C*. *CE* and *AD* meet at *B*.
Prove that angle *ACB* = 2(angle *DEB*).



Question 3 (Start a new page). (12 marks).

The polynomial $P(x) = 0$ has a double root at $x = a$.

a) By putting $P(x) = (x-a)^2Q(x)$, show that $P'(a) = 0$. **(2)**

b) The equation $P(x) = mx^4 + nx^3 - 6x^2 + 22x - 12 = 0$ has a double root at $x = 1$. Find the values of m and n. **(2)**

c) Using the fact that $x = 3$ is also a root of $P(x)$, express $P(x)$ in factorised form. **(2)**

d) Solve $P'(x)$=0 and state if $x = 1$ represents a local minimum or local maximum. **(2)**

e) Show on a sketch of $P(x)$ where it cuts the axes and the x-values of the turning points. **(3)**

f) From the graph solve $P(x) > 0$ **(1)**

Question 4 (Start a new page). (12 marks).

a) Let $S_n = 1^2 + 2^2 \dots\dots\dots + n^2$, for $n = 1,2,3,\dots$

i) Use Mathematical Induction to prove that for $n = 1,2,3,\dots$ **(4)**

$$S_n = \frac{1}{6}n(n+1)(2n+1)$$

ii) By using the result of part (i) estimate the least n such that $S_n \geq 10^{11}$ **(2)**

b) i) How many different arrangements are there of the letters ARRANGEMENT **(1)**

ii) How many of these begin with the letter R? **(2)**

c) Evaluate $\int_0^{\frac{\pi}{4}} tan^3xsec^4x\, dx$, using the substitution $u = tanx$ **(3)**

Question 5 (Start a new page). **(12 marks).**

a) Evaluate $\int_0^{\frac{\pi}{2}} cos^2 3x\, dx$ **(3)**

b) i) Express $-sinx - \sqrt{3}cosx$ in the form $Rcos(x - \alpha)$ **(3)**

ii) Hence solve $-sinx - \sqrt{3}cosx = 2$ for $0 \leq x \leq 2\pi$. **(3)**
Give the solutions correct to 3 decimal places.

c) Prove that $\frac{1-tan\theta tan2\theta}{1+tan\theta tan2\theta} = 4cos^2\theta - 3$ **(3)**

Question 6 (Start a new page). **(12 marks).**

$a)$ $f(x) = x - \frac{1}{x}, x > 0$

i) Show that $f(x)$ has no stationary points. **(1)**

ii) Describe the behaviour of $f(x)$ as x approaches the extremities of its domain. **(1)**

iii) Sketch, on one diagram, graphs of $y = x,\ y = f(x)\ and\ y = f^{-1}(x)$ **(3)**

iv) If $x = y - \frac{1}{y}$ and $y > 0$, simplify, in terms of y, $x + \sqrt{x^2 + 4}$ **(2)**

v) Write the expression for $f^{-1}(x)$ in terms of x **(1)**

b) i) Find $\frac{d}{dx}[sin^{-1}(2x - 1)]$ **(2)**

ii) Hence, deduce that $\int_{\frac{3}{4}}^{1} \frac{dx}{\sqrt{x-x^2}} = \frac{\pi}{3}$ **(2)**

Question 7 (Start a new page). **(12 marks).**

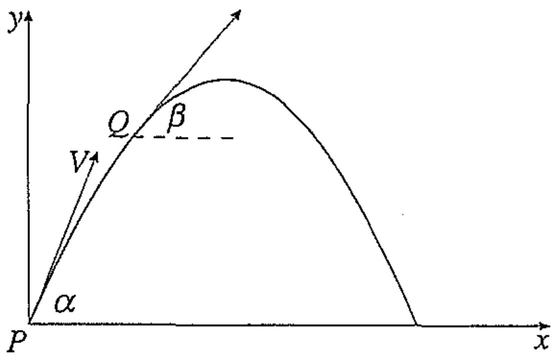
a) The rate at which a body cools is assumed to be proportional to the difference between its temperature T and the constant temperature P of the surrounding medium.
This can be expressed by the differential equation:

$\frac{dT}{dt} = k(T - P)$ where t is the time in hours and k is a constant.

i) Show that $T = P + Ae^{kt}$, where A is a constant, is a solution of the differential equation. **(1)**

ii) A heated body cools from 90°C to 50°C in 3 hours. The temperature of the surrounding medium is 20°C. Find the temperature of the body after one further hour has elapsed. Give your answer correct to the nearest degree. **(3)**

b) A particle is projected from a point P on horizontal ground, with speed Vms^{-1} at an angle of elevation to the horizontal of α.



Its equations of motion are $\ddot{x} = 0$, $\ddot{y} = -g$.

i) Write down expressions for its horizontal (x) and vertical (y) displacements from P after t seconds. **(2)**

ii) Determine the time of flight of the particle. **(1)**

iii) The particle reaches a point Q, as shown, where the direction of flight makes an angle β with the horizontal. Show that the time taken to travel from P to Q is: **(2)**

$$\frac{V\sin(\alpha-\beta)}{g\cos\beta} \text{ seconds.}$$

iv) Consider the case when $\beta = \frac{\alpha}{2}$. If the time taken to travel from P to Q is then one-third of the total time of flight, find the value of α. **(3)**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n\neq -1;\ x\neq 0,\ if\ n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\ a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a\neq 0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln(x+\sqrt{x^2-a^2}),\ x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln(x+\sqrt{x^2+a^2})$$

NOTE: $\ln x = \log_e x,\ x>0$

Q1 a) $(-3,5)$ $(8,4)$ $\quad 1:3$ $\quad m:n$

$$x = \frac{3\times -3 + 1\times 8}{4} = -\frac{1}{4} \qquad y = \frac{3\times5 + 1\times 4}{4} = \frac{19}{4}$$

$$\left(-\tfrac{1}{4}, \tfrac{19}{4}\right)$$

b) $y = 8 - 3x$, $\quad m_1 = -3$

$$\tan 45^\circ = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|,$$

$$1 = \left|\frac{-3 - m_2}{1 - 3m_2}\right|$$

$$\frac{-3 - m_2}{1 - 3m_2} = 1 \quad \text{or} \quad \frac{-3 - m_2}{1 - 3m_2} = -1$$

$-3 - m_2 = 1 - 3m_2 \qquad\qquad -3 - m_2 = 3m_2 - 1$

$2m_2 = 4 \qquad\qquad -2 = 4m_2$

$m_2 = 2 \quad$ or $\quad m_2 = -\tfrac{1}{2}$

At $(4,0)$ $\quad y - 0 = 2(x-4) \qquad\qquad y = -\tfrac{1}{2}(x-4)$

$y = 2x - 8 \qquad\qquad y = -\tfrac{1}{2}x + 2$

c) $\int x\sqrt{3-x}\,dx \qquad u = 3 - x$ So $x = 3 - u$

$\frac{du}{dx} = -1$ So $dx = -du$

$$= \int u^{\frac{1}{2}}(3-u)\,{-du}$$

$$= \int -3u^{\frac{1}{2}} + u^{3/2}\,du = \frac{2}{5}u^{5/2} - 2u^{3/2} + C$$

$$= \frac{2}{5}\sqrt{(3-x)^5} - 2\sqrt{(3-x)^3} + C$$

d) $\frac{2x+5}{x} \geq 1 \qquad$ Case 1 $\quad x > 0 \qquad$ Case 2 $\quad x < 0$

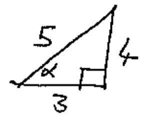
$2x + 5 \geq x \qquad\qquad x \leq -5$

$x \geq -5 \qquad\qquad \therefore \boxed{x \leq -5}$

$\therefore \boxed{x > 0}$ → or ←

1e) $\sin\left[\cos^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(\frac{-3}{4}\right)\right] = \frac{1}{25}$ ①

Let $\alpha = \cos^{-1}\left(\frac{3}{5}\right)$



$\sin\alpha = \frac{4}{5}$

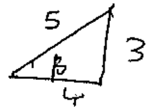
Let $\beta = \tan^{-1}\left(\frac{-3}{4}\right) \quad -\frac{\pi}{2} < \beta < \frac{\pi}{2}$ Range of $\tan^{-1}$

$\therefore$ 4th quad

S | A
T | C

$\cos\beta = \frac{4}{5}$

$\sin\beta = -\frac{3}{5}$



LHS of ① $= \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \frac{4}{5}\times\frac{4}{5} + \frac{3}{5}\times\frac{-3}{5} = \frac{7}{25}$$

Q2) a)

$x = \sin\theta + \cos\theta \quad y = \sin\theta - \cos\theta$

$x^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$

$y^2 = \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta$.

circle centre (0,0) radius $\sqrt{2}$

$x^2+y^2 = 2(\sin^2\theta + \cos^2\theta)$

$x^2+y^2 = 2$ $\therefore$ circle, centre (0,0) radius $= \sqrt{2}$

b) i) $x^2 = 4ay$ at $(2ap, ap^2)$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$.

when $x = 2ap$, $\frac{dy}{dx} = \frac{2ap}{2a} = p$.

$\therefore$ Gradient of normal $= -\frac{1}{p}$

$\therefore$ Eqn of Normal

$y - ap^2 = -\frac{1}{p}(x - 2ap)$.

$py - ap^3 = -x + 2ap$.

$x + py = ap^3 + 2ap$.

ii) Chord $P(2ap, ap^2)$ + $Q(2aq, aq^2)$

$m = \frac{ap^2 - aq^2}{2ap - 2aq} = \frac{a(p+q)(p-q)}{2a(p-q)}$

$= \frac{p+q}{2}$

Eqn of Chord

$y - ap^2 = \frac{p+q}{2}(x - 2ap)$.

$y - ap^2 = \frac{p+q}{2}(x) - ap^2 - apq$.

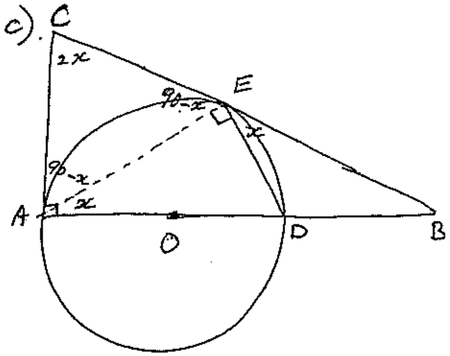
$y = \frac{p+q}{2}(x) - apq$

now sub $T(0, -2a)$.

$-2a = \frac{p+q}{2}(0) - apq$.

$2 = pq$

c)



Let $\angle DEB = x$.

$\angle EAD = x$ ($\angle$ b/t tang. + chord = $\angle$ in alt seg)

$\angle CAD = 90°$ ($\angle$ b/t tang + radius)

$\therefore \angle CAE = 90° - x$.

$\angle AED = 90°$ ($\angle$ in semi-circle)

$\therefore \angle CEA = 180 - 90 - x$ (CEB collinear)

$= 90 - x$

$\therefore$ In $\triangle ACE$

$\angle ACE = 180 - 2(90 - x)$

$= 180 - 180 + 2x$

$= 2x$.

$\therefore \angle ACE = 2 \times \angle DEB$

Q3

a) $P(x) = (x-a)^2 Q(x)$

$P'(x) = (x-a)^2 Q'(x) + 2(x-a)Q(x)$

$P'(a) = (a-a)^2 Q'(a) + 2(a-a)Q(a)$

$= 0^2 Q'(a) + 2(0)Q(a)$

$= 0$

b) $P(x) = mx^4 + nx^3 - 6x^2 + 22x - 12$

$P'(x) = 4mx^3 + 3nx^2 - 12x + 22$

$P(1) = m + n - 6 + 22 - 12 = 0$

$m + n = -4$ ---(1)

$P'(1) = 4m + 3n - 12 + 22 = 0$

$4m + 3n = -10$ ...(2)

$\times$(1) by 3

From $3m + 3n = -12$ ...(3)

(2) - (3) $m = 2$.

Sub into (1)

$2 + n = -4$

$n = -6$

$\therefore m = 2 \quad n = -6$

c) $P(x) = 2x^4 - 6x^3 - 6x^2 + 22x - 12 = 0$

$(x-1)^2(x-3)$ is a factor

ie $(x^2 - 2x + 1)(x - 3)$

$= x^3 - 5x^2 + 7x - 3$

$$\begin{array}{r} 2x + 4 \\ x^3 - 5x^2 + 7x - 3 \enclose{longdiv}{2x^4 - 6x^3 - 6x^2 + 22x - 12} \\ \underline{2x^4 - 10x^3 + 14x^2 - 6x} \\ 4x^3 - 20x^2 + 28x - 12 \\ \underline{4x^3 - 20x^2 + 28x - 12} \end{array}$$

$\therefore P(x) = (x-1)^2(x-3)(2x+4)$

$= 2(x-1)^2(x-3)(x+2)$

d) $P'(x) = 8x^3 - 18x^2 - 12x + 22$.

$P'(x) = 0$

$x - 1$ is a factor as $x = 1$ is a double root.

$$\begin{array}{r} 8x^2 - 10x - 22 \\ x - 1 \enclose{longdiv}{8x^3 - 18x^2 - 12x + 22} \\ \underline{8x^3 - 8x^2} \\ -10x^2 - 12x \\ \underline{-10x^2 + 10x} \\ -22x + 22 \\ \underline{-22x + 22} \end{array}$$

$\therefore P'(x) = (x-1)(8x^2 - 10x - 22)$

$= 2(x-1)(4x^2 - 5x - 11)$

$P'(x) = 0$

$\therefore x - 1 = 0$ or $4x^2 - 5x - 11 = 0$

$x = 1$ or $x = \frac{5 \pm \sqrt{201}}{8}$

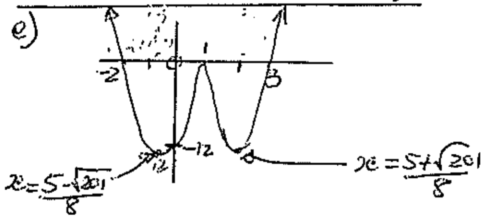
$P''(x) = 24x^2 - 36x - 12$.

$P''(1) = 24 - 36 - 12$.

$= -24$

$\therefore P''(1) < 0 \therefore$ local max at $(1, 0)$

e)



f) $P(x) > 0$

$x < -2, \; x > 3$

4. a) i) Prove $1^2+2^2+\dots+n^2 = \frac{1}{6}n(n+1)(2n+1)$ ①

Step 1 Prove statement true for $n=1$

LHS of ① $=1^2=1$ RHS of ① $=\frac{1}{6}\times1\times2\times3=1$

LHS = RHS ∴ Statement true for $n=1$.

Step 2 Assume statement true for $n=k$

ie assume $1^2+2^2+\dots+k^2=\frac{1}{6}k(k+1)(2k+1)$

Prove statement true for $n=k+1$

ie prove $1^2+\dots+k^2+(k+1)^2=\frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$

$=\frac{1}{6}(k+1)(k+2)(2k+3)$ ②

Proof: LHS of ② $=\frac{1}{6}k(k+1)(2k+1)+(k+1)^2$ (By assumption)

$=\frac{1}{6}(k+1)(k(2k+1)+6(k+1))$

$=\frac{1}{6}(k+1)(2k^2+k+6k+6)$

$=\frac{1}{6}(k+1)(2k^2+7k+6)$

$=\frac{1}{6}(k+1)(k+2)(2k+3)$

= RHS of ②.

∴ Assuming statement true for $n=k$ we have proved statement true for $n=k+1$.

Step 3 Assuming statement true for $n=k$ then statement true for $n=k+1$. Statement has been proved true for $n=1$, ∴ statement true for $n=1+1=2$, ∴ statement true for $n=2+1=3$, etc ∴ By mathematical induction, statement true for all positive integers n.

(ii) Solve $S_n \geq 10^{11}$

∴ $\frac{1}{6}n(n+1)(2n+1) \geq 10^{11}$

$n(n+1)(2n+1) \geq 6\times10^{11}$

$n(2n^2+3n+1) \geq 6\times10^{11}$

$2n^3+3n^2+n \geq 6\times10^{11}$

($2n^3$: Dominant term)

$\sqrt[3]{3\times10^{11}}=6694.3$ Try 6694: 6.0004×10^{11}; 6693: 5.9977×10^{11}

∴ $n=6694$ is least.

b) i) $\frac{11!}{2!2!2!2!}=2494800$

(2!2!2!2!: R's, A, N, E swapping places)

ii) $\frac{10!}{2!2!2!}$ or $\frac{2\times10!}{2!2!2!2!}$ (Place R)

$=453600$

c) $\int_0^{\frac{\pi}{4}} \tan^3x\sec^4x\,dx$

$u=\tan x$

$\frac{du}{dx}=\sec^2x$

$\frac{du}{\sec^2x}=dx$

$\sec^2x=1+\tan^2x=1+u^2$

$x=0,\ u=\tan0=0$

$x=\frac{\pi}{4},\ y=\tan\frac{\pi}{4}=1$

$=\int_0^{\frac{\pi}{4}} u^3\sec^2x\,du$

$=\int u^3(1+u^2)\,du$

$=\int_0^1 u^3+u^5\,dx$

$=\left[\frac{u^4}{4}+\frac{u^6}{6}\right]_0^1$

$=\frac{1}{4}+\frac{1}{6}-0=\boxed{\frac{5}{12}}$

Q5)

a) $\int_0^{\frac{\pi}{2}} \cos^2 3x \, dx$.

$= \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 6x)\, dx$

$= \frac{1}{2}\left[x + \frac{\sin 6x}{6}\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left\{\left(\frac{\pi}{2} + \frac{\sin 3\pi}{6}\right) - 0\right]$

$= \frac{1}{2}\left[\frac{\pi}{2} + 0\right]$

$= \frac{\pi}{4}$

b). $-\sin x - \sqrt{3}\cos x$.

$R\cos(x-\alpha) = R(\cos x \cos\alpha + \sin x \sin\alpha)$

$R = \sqrt{1^2+\sqrt{3}^2} = \sqrt{4} = 2$

$\therefore -\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \cos x \cos\alpha + \sin x \sin\alpha$

where $\cos\alpha = -\frac{\sqrt{3}}{2}$ $\sin\alpha = -\frac{1}{2}$

$\tan\alpha = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$ } 3rd quad.

$\alpha = \frac{7\pi}{6}$.

$\therefore -\sin x - \sqrt{3}\cos x = 2\cos\left(x - \frac{7\pi}{6}\right)$

(OR) $= -2\cos\left(x - \frac{\pi}{6}\right)$

ii). $2\cos\left(x - \frac{7\pi}{6}\right) = 2$

$\cos\left(x - \frac{7\pi}{6}\right) = 1$

$x - \frac{7\pi}{6} = 0, 2\pi$.

$x = 0 + \frac{7\pi}{6}, 2\pi + \frac{7\pi}{6}$

$\therefore x = \frac{7\pi}{6}$, $0 \le x \le 2\pi$

$\doteqdot 3.665$

c) $\frac{1 - \tan\theta\tan 2\theta}{1 + \tan\theta\tan 2\theta}$.

$= \frac{1 - \tan\theta\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)}{1 + \tan\theta\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)}$

$= \frac{\frac{1-\tan^2\theta - 2\tan^2\theta}{1-\tan^2\theta}}{\frac{1-\tan^2\theta + 2\tan^2\theta}{1-\tan^2\theta}}$

$= \frac{1-3\tan^2\theta}{1+\tan^2\theta}$.

$= \frac{1 - \frac{3\sin^2\theta}{\cos^2\theta}}{1 + \frac{\sin^2\theta}{\cos^2\theta}}$.

$= \frac{\cos^2\theta - 3\sin^2\theta}{\cos^2\theta + \sin^2\theta}$.

$= \cos^2\theta - 3(1-\cos^2\theta)$

$= 4\cos^2\theta - 3$

Q6

a)(i) $f(x) = x - \frac{1}{x}$ $x > 0$

$f(x) = x - x^{-1}$

$f'(x) = 1 + x^{-2}$

$= 1 + \frac{1}{x^2}$

$f'(x) = 0$ ← stationary pts.

$1 + \frac{1}{x^2} = 0$

$\frac{1}{x^2} = -1$

$1 = -x^2$

$-1 = x^2$

as $x^2 \neq -1$

$\therefore$ no solution

$\therefore$ no stationary points

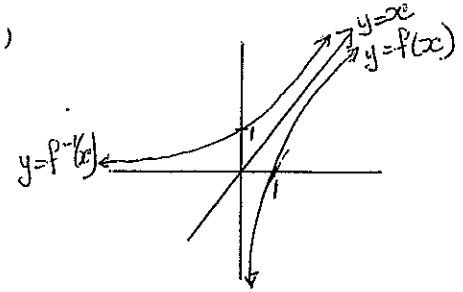
(ii) As $x \to \infty$ $\frac{1}{x} \to 0$

$\therefore x - \frac{1}{x} \longrightarrow x$ (from below)

As $x \to 0$ $\frac{1}{x} \to \infty$

$\therefore x - \frac{1}{x} \longrightarrow -\infty$

(iii)



(iv) $x = y - \frac{1}{y}$

i.e.

$x + \sqrt{x^2+4} = y - \frac{1}{y} + \sqrt{\left(y-\frac{1}{y}\right)^2+4}$

$= y - \frac{1}{y} + \sqrt{y^2 - 2 + \frac{1}{y^2} + 4}$

$= y - \frac{1}{y} + \sqrt{y^2 + 2 + \frac{1}{y^2}}$

iv)

$= y - \frac{1}{y} + \sqrt{\left(y+\frac{1}{y}\right)^2}$

$= y - \frac{1}{y} + y + \frac{1}{y}$

$= 2y$

v) $f(x): y = x - \frac{1}{x}$

$f^{-1}(x): x = y - \frac{1}{y}$

$\therefore 2y = x + \sqrt{x^2+4}$

$y = \frac{x + \sqrt{x^2+4}}{2}$

b)(i) $\frac{d}{dx}\left[\sin^{-1}(2x-1)\right]$

ie

$= \frac{d}{dx}\left[\sin^{-1} u\right]$ where $u = 2x-1$

$= \frac{1}{\sqrt{1-u^2}} \times \frac{d}{dx}(2x-1)$

$= \frac{1}{\sqrt{1-(2x-1)^2}} \times 2$

$= \frac{2}{\sqrt{1-(4x^2-4x+1)}}$

$= \frac{2}{\sqrt{-4x^2+4x}}$

$= \frac{2}{\sqrt{4x-4x^2}}$ $= \frac{2}{\sqrt{4(x-x^2)}}$

$= \frac{1}{\sqrt{x-x^2}}$

(ii) $\int_{\frac{3}{4}}^{1} \frac{dx}{\sqrt{x-x^2}}$ $= \left[\sin^{-1}(2x-1)\right]_{\frac{3}{4}}^{1}$

$= \sin^{-1}(1) - \sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{2} - \frac{\pi}{6}$

$= \frac{\pi}{3}$

$\therefore$ = RHS.

7\. a) $T = P + Ae^{kt}$ so $Ae^{kt} = T - P$ ①

$\frac{dT}{dt} = Ae^{kt} \times k$

$= k(T-P)$ from ①

$\therefore$ $T = P + Ae^{kt}$ a solution of $\frac{dT}{dt} = k(T-P)$

(ii) $t=0,\ T=90$ $\quad t=3,\ T=50 \quad P=20$

Find T when $t=4$.

$T = P + Ae^{kt}$

$t=0,\ T=90,\quad 90 = 20 + Ae^{0}$

$\therefore A = 70$

$\therefore T = 20 + 70e^{kt}$

$t=3,\ T=50 \quad \therefore 50 = 20 + 70e^{3k}$

$30 = 70e^{3k}$

$3/7 = e^{3k}$

$\ln(\frac{3}{7}) = 3k$

$\therefore k = \frac{1}{3}\ln(3/7) \qquad \therefore T = 20 + 70e^{\frac{1}{3}\ln k t}$

$t=4$ $\qquad T = 20 + 70e^{(\frac{1}{3}\ln k)\times 4}$

$T = 43^\circ$ (to nearest degree)

b) (i) $\ddot{x} = 0$

$\dot{x} = C_1$

$t=0 \quad \dot{x} = V\cos\alpha \ \therefore C_1 = V\cos\alpha$

$\therefore \dot{x} = V\cos\alpha$

$x = Vt\cos\alpha + C_2$

$t=0, x=0 \ \therefore C_2 = 0$

$\therefore x = Vt\cos\alpha$

$\ddot{y} = -g$

$\dot{y} = -gt + C_3$

$t=0,\ \dot{y} = V\sin\alpha$

$\therefore C_3 = V\sin\alpha$

$\therefore \dot{y} = -gt + V\sin\alpha$

$y = \frac{-gt^2}{2} + Vt\sin\alpha + C_4$

$t=0, y=0, \ \therefore C_4 = 0$

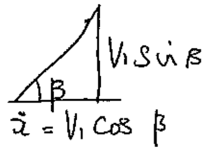
$\therefore y = \frac{-gt^2}{2} + Vt\sin\alpha$

(ii) Time of flight $y=0$

$\therefore t(V\sin\alpha - \frac{gt}{2}) = 0 \qquad t=0$ (start)

$\therefore V\sin\alpha = \frac{gt}{2} \qquad \therefore \boxed{t = \frac{2V\sin\alpha}{g}}$

(iii)



$\dot{y}$ $\qquad \tan\beta = \frac{V_1\sin\beta}{V_1\cos\beta} = \frac{\dot{y}}{\dot{x}}$

Also $\frac{\dot{y}}{\dot{x}} = \frac{V\sin\alpha - gt}{V\cos\alpha}$ $\quad$ where t is time from P to Q.

$\therefore \frac{\sin\beta}{\cos\beta} = \frac{V\sin\alpha - gt}{V\cos\alpha}$

$V\sin\beta\cos\alpha = \cos\beta(V\sin\alpha - gt)$

$V\sin\beta\cos\alpha = V\sin\alpha\cos\beta - gt\cos\beta$

$\therefore gt\cos\beta = V\sin\alpha\cos\beta - V\cos\alpha\sin\beta$

$= V\sin(\alpha-\beta)$

$\therefore \boxed{t = \frac{V\sin(\alpha-\beta)}{g\cos\beta}}$

(iv) $\frac{V\sin(\alpha-\beta)}{g\cos\beta} = \frac{1}{3}\left(\frac{2V\sin\alpha}{g}\right) \rightarrow$ time of flight

$\beta = \frac{\alpha}{2} \quad \therefore \frac{V\sin(\alpha - \frac{\alpha}{2})}{g\cos\alpha/2} = \frac{2}{3}\frac{V\sin\alpha}{g}$

$\frac{\sin\alpha/2}{\cos\alpha/2} = \frac{2}{3}\sin\alpha$

$\frac{\sin\alpha/2}{\cos\alpha/2} = \frac{2}{3}(2\sin\alpha/2\cos\alpha/2)$ Double angle

$3\sin\alpha/2 = 4\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\cos\frac{\alpha}{2}$

$\div \sin\alpha/2$

$4\sin\alpha/2\cos^2\alpha/2 - 3\sin\alpha/2 = 0$

$\sin\alpha/2(4\cos^2\alpha/2 - 3) = 0$

$\sin\alpha/2 = 0 \qquad \frac{\alpha}{2} = 0$ ✗

$\therefore \cos^2\alpha/2 = \frac{3}{4} \quad \cos\frac{\alpha}{2} = \frac{\sqrt{3}}{2} \qquad \frac{\alpha}{2} = \frac{\pi}{6}$

$\boxed{\alpha = \pi/3}$